Prerequisities

In this book there are addressed issues considered in two fields of mathematics (or, one can say, of applications of mathematics); these fields are the probability theory (both in its classical meaning and as the Kolmogorov system) and the mathematical statistics. Problems treated by them arise from an every day observation and require a quite good orientation in recurrences for elements forming numerical sequences, in functions generating such sequences, in differential and integral calculus (of functions in one and many variables), even in differential equations. To standardize the notions and symbols, as well as to recall some basic definitions and procedures, or to present them to a reader, we run through selected items. They include a little bit of the set theory (with the stress on fundamental logic, the equinumeracy, defined via a bijection, and the countableness and the uncountability of sets), of the recursion theory (presented via the Fibonacci sequence).

On an axiomatic structure of the mathematics

The probability theory and the mathematical statistics are branches of mathematics. The mathematics is developed "since ever" (although it started to take its contemporary form around 300 BC, when Euclid presented his *Elements*). It turned to be a very wide science (which still remains the only one dealing with the infinity), it is present almost everywhere (in particular in its basic applications: in the enumerating and the counting), its importance grows as (and it is clearly seen in last decades when the world experiences the digitalization, which is manifested in such devices as computers, CDs and digital photo cameras). Probably it is all impossible to define what it is the mathematics. Nevertheless, we can say that the mathematics is this area of human mental activity which arises from reflections on numbers and geometrical figures, which generalizes these notions.

The mathematics (as well as such theoretical computer science, system theory, formal linguistics) is a formal science. This says that it is interested in the characterization of abstract structures, while natural sciences (empirical physics, chemistry, biology etc.) are interested in the description of physical systems (i.e., systems existing, or potentially existing, in the physical world, where every object takes a room and weights or, at least, is detected via measuring instruments).

Contemporary mathematics is an axiomatic science. This means that it is a theory (instead of 'theory' one can say 'description') constructed on primitive notions, axioms and rules; without going in details we can say that

a) **primitive notions** (instead of 'notion' one can say 'object', 'concept') are objects existing by themselves,
b) **axioms** (instead of 'axiom' one can say 'assumption') are statements (instead of 'statement' one can say 'claim', 'theorem') assumed to be true,
c) **rules** (more precisely: logical rules) are recipes according to whose new objects can be defined (instead of 'define' one can say 'introduce') and

new statements can be derived (instead of ‘derive’ one can say ‘deduce’, ‘prove’, ‘formulate’).

One can say that the mathematics has the fully analogous structure to the institutional faith, such as realized as Roman Catholic church: without hurting anyone's religious feelings there are here primitive objects (God), axioms (called dogmas; their examples: God is infinitely righteous; God exists in three hypostases; the Virgin Mary was conceived immmaculate – declared in 1854 by Pope Pius IX in his bull *Ineffabilis Deus*) and rules (declared as Ten Commandments, i.e., in the Decalogue).

On mathematical logic

In mathematics there are observed the logical rules, and this branch of mathematics which investigates them and explores their applications is called (after Giuseppe Peano's proposal) a **mathematical logic**. It is a formal science, usually it used small letters to denote logic variables (i.e., variables which assumes value either true or false [1]), special symbols for logic operations (they are negation, conjunction and disjunction, aka logical product and logical sum, resp., product, implication and equivalence, commonly denoted by $\sim$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$, resp.) and provides tautologies (i.e., sentences which remains true for any values of their arguments). It is enough to take, for instance, the conjunction and the negation as primitive operations, and then all other operations are defined. Examples of these rules are given in Table on next page.

Several logical rules were worked out in the antiguity.
First attempts to treat algebraically the logic were undertaken by G.W.Leibniz in 1690s, he introduced such properties as conjunction, disjunction, negation, set inclusion, he formulated such fundamental principle as the following one: two sets are equal iff they have the same elements.
Among numerous mathematicians investigating considered problems let's list here J.W.Lambert (about 1750), Augustus de Morgan (he rediscover today called de Morgan laws in 1850), George Boole (his *The laws of thought* were published in 1854) and Georg Cantor, with the last one a modern set theory starts, in his *Über unendliche, lineare Punktmannichfaltigkeiten* (1883) for the first time appears the term 'set theory' (in its German sound: *Mengelehre*).

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[1] Here we deal a little with two-valued (or bivalent, binary) logic. In this logic every declarative sentence (expressing a proposition of a theory under inspection) has one of two possible values: true and false. This fact, equivalent to the law of the excluded middle, is known as a principle of bivalence and was clearly stated by Aristotle (and two-valued logic is aka Aristotlean logic). Among problems considered by Aristotle it is 'future contingents proposition' (this is a statement about the future that is neither necessarily true nor necessarily false), illustrated via the sea-battle example (either there will be a battle or there won't, both options can't be taken at once, and today we can it is impossible to not say which one option is correct, we must wait and after some time it will take a correct value – 'the logic will realize itself'). This problem was treated, a.o., by G.W.Leibniz, Immanuel Kant, Jan Łukasiewicz and Stephen Cole Kleene. Łukasiewicz, a Polish logician and philosopher, in 1917 developed a three-valued propositional calculus where there are in use three truth values: true, false and some indeterminate third value (one can think of it as 'unknown' or 'maybe'). Kleene (he, Alan Turing and Emil Post were student of Alonzo Church) in 1952 proposed another three-valued logic (it differs from Łukasiewicz's one by the table of implication). Let's mention that in 1938 Kleene presented theorems on recursion, in 1920 Łukasiewicz worked out prefix Polish notation, both achievements play fundamental role in computer science.

Table. Some rules (tautologies) in mathematical logic

$\sim\sim p \Leftrightarrow p$ – double negation principle,

$p \vee \sim p$ – law of the excluded middle,

$\sim(p \wedge \sim p)$ – law of non-contradiction,

$\sim(a \vee b) \Leftrightarrow (\sim a \wedge \sim b)$,

$\sim(a \wedge b) \Leftrightarrow (\sim a \vee \sim b)$ – de Morgan laws, named after Augustus de Morgan who presented them in the paper *On the symbols of logic, the theory of the syllogism, and in particular of the copula* (1850); similar observation was made by Aristotle (c.330 BC) and applied by William Ockham in his *Summa totius logicae* (1341),

$(a \Rightarrow b) \Leftrightarrow (\sim a \vee b)$ – ind (the implication, negation, disjunction),

$(a \Rightarrow b) \Leftrightarrow \sim(a \wedge \sim b)$ – inc (the implication, negation, conjunction),

$\{ (p \Rightarrow q) \wedge p \} \Rightarrow q$ – modus ponens rule,

$\{ (p \Rightarrow q) \wedge \sim q \} \Rightarrow \sim p$ – modus tollens rule,

$\{ (p \Rightarrow q) \wedge \sim q \} \Rightarrow \sim p$ – modus tollendo rule,

$\sim p \Rightarrow (p \Rightarrow q)$ – Duns Scotus law; Duns Scotus was an eminent mediaval philosopher living at the turn of the 13th century,

$(\sim p \Rightarrow p) \Rightarrow p$ – Clavius tautology; Clavius lived in the late 16th century and commented Euclid's *Elements*,

$((a \Rightarrow (b \Rightarrow c)) \wedge (a \Rightarrow b)) \Rightarrow (a \Rightarrow c)$ – Frege schema; Gottlob Frege presented the propositional logic as a formalized axiomatic system in his book *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (*Concept Notation: A formula language of pure thought, modelled upon that of arithmetic* (*1879*)

$\{ p \Rightarrow q \} \Leftrightarrow \{ \sim q \Rightarrow \sim p \}$ – the contraposition law,

$(\sim a \Rightarrow (b \wedge \sim b)) \Rightarrow a$ – the reduction ad absurdum principle.

The first axiomatic theory was provided by Euclid. It concerned the geometry, today known as an **Euclidean geometry** (and it took its final form as late as in 1899 in David Hilbert's book *Grundlagen der Geometrie* [2]). In Euclid's systematization the primitive objects are point, straight line and plane, as well as the betweeness, the containment and the congruence (first two of them are relations between primitive objects, the third one relates line segments and angles, which are already secondary objects, i.e., objects defined in the system). Euclid assumed only five axioms, the last one of them states that through a given point laying out of a given line there passes exactly one line parallel to that given. With this axiom omitted we have so-called **absolute geometry**, and we have so-called **elliptic geometry** (aka **Riemann geometry**) and **hyperbolic geometry** (aka **Bolyai-Lobachevski geometry**) when it is replaced by the assumption that there is no parallel line and there are infinitely many parallel lines, resp. Both last conditions are equivalent to the assumption that in every flat triangle the sum of angles is greater than 180° and is less than 180°, resp.
Since ever it also deal with sets (instead of 'set' one can say 'collection') such as sets of numbers, sets of triangles, sets of circles.

[2] Hilbert originally proposed 21 axioms. In 1902 E.Moore and R.Moore independently showed that one axiom is redundant. Other modern axiomatizations of Euclidean geometry were provided by George Birkhoff in his *A set of postulates for plane geometry (based on scale and protractors)*, 1932, and by Alfred Tarski (the first approach is reported in his paper *What is elementary geometry ?*, 1959).

A bit of the set theory

One can say that the mathematics is this area of human mental activity which arises from reflections on numbers and geometrical figures. Since ever it also deal with sets (instead of 'set' one can say 'collection of objects') such as sets of numbers, sets of triangles, sets of circles. But only in the second half of 19th century the sets became the subject of extensive research. Deep studies began with *Paradoxien des Unendlichen* (Paradoxes of the infinity, 1851) by Bernard Bolzano and with *Über eine Eigenschaft des Inbegriffes aller algebraischen Zahlen* (*On a characteristic property of all real algebraic numbers*, 1874) by Georg Cantor. Bolzano and Cantor, as well as R.Dedekind, K.Weierstrass, H.Lebesgue, B.Russell and many others made that the set theory was mounted to be a fundamental branch of mathematical logic and mathematics itself. All the mathematics can be built on the set-theoretical fundaments, and in this mathematics

- its primitive notions are, a.o., a set, a relation 'to be an element of a set',
- its axioms are axioms of the set theory – such as **axiom of pairing** (it says that for any two sets, A and B, there exists a set whose elements are exactly A and B),
- there are defined such objects as
  a) a **pair**: $(a, b) := \{ a, \{ b \} \}$,
  b) a **Cartesian product** of sets A and B:
  $$A \times B := \{ (a, b): a \in A; b \in B \},$$
  c) a **relation** in a Cartesian product; by definition, it is any subset of this product,
  d) an **equivalence** (**relation**) in a set A; by definition, it is a relation in $A \times A$, usually denoted by $\sim$, which is
  reflexive, i.e., $a \sim a$,
  symmetric, i.e., $a \sim b \Rightarrow b \sim a$,
  and **transitive**, i.e., $\{ (a \sim b) \wedge (b \sim c) \} \Rightarrow (a \sim c)$,
  e) a **function** (or a **map**(**ping**), a **transformation**) from X to Y; by definition, it is a relation f in $A \times V$ satisfying the implication:
  $$\{ y = f(a) \wedge y = f(x) \} \Rightarrow (a = x),$$
  where $v = f(a)$ is read 'v is a value of the function f at a (or: the function f assumes the value v for the argument equal to a)' written instead of $a\,f\,v$; commonly:
  - we write then $f: A \to V$,
  - the set A the arguments of the function f are taken from is called the **domain** of f,
  - the set $f(A) := \{ f(a): a \in A \}$ is called a f-**image** of A, or a **codomain** of f, and is sometimes denoted as $\mathrm{Im}\, f$,
  - the set $\{ a \in A : f(a) \in B \}$ is called an **inverse image**, or a **preimage** (under the function f), of the set B and is denoted as $f^{-1}(B)$.

Today the set theory is grounded on Zermelo-Fraenkel systematization (devised by Ernst Zermelo in 1908, and completed by Abraham Fraenkel in 1921).

In general, the dealing with

- finite sets is easy (the theory concerning finite sets is fully covered by so-called **naïve set theory**, and it is well illustrated with so-called Venn diagrams),
- countable sets it is more complex (because there appear paradoxes such as Hilbert hotel, where there can be found infinitely many free rooms even when there are already all occupied),
- and, in many cases, it is really difficult with uncountable sets (with the axiom of choice it implies such phenomena as Tarski-Banach paradox, where a solid ball in the space $\boldsymbol{R}^3$ can be decomposed into two identical balls).

This course is offered to engineering students, so, as far as it is possible, it will not go in the intricacies. Nevertheless, the reader has to be conscious of the complexities. Moreover, he/she will meet them when there will be discussed the problem posed by Joseph Bertrand (it results in so-called Bertrand paradox). The troubles like this made that it was worked, in 1933 by A.Kolmogorov, out so-called axiomatic theory of probability.

By the tradition, power functions, trigonometric functions and their inverses, logarithmical and exponential functions, as well as the sums, differences, products and quotients of these all functions, are classified as **elementary functions**. Examples of elementary functions are the maps which transform x into

a) x^2 (called a **square** of x, or x raised to the 2nd power) and $x^{1/2}$ (called a **square root** of x, or x raised to the exponent 1/2),
b) a **polynomial** (it has form $c_0 + c_1 \cdot x + \ldots + c_n \cdot x^n$, where $c_0, c_1, \ldots, c_n$ are given numbers and are called its coefficients),
c) a **rational function** (by definition, it is the quotient of two polynomials), e.g., a logistic function (introduced by
d) $\arcsin x$ (read 'arcus sine of x', said to be the inverse of the sine, or an inversed sine),
e) $\operatorname{arsinh} x$ (read as 'area hyperbolic sine of x', said to be the inverse of the hyperbolic sine, or an inversed hyperbolic sine).

Any function which is not elementary is said to be **non-elementary**. Basic non-elementary functions are

a) gamma function (it is commonly denoted by Γ, it extends the factorial),
b) error function (it is denoted by Erf),
c) integral sines and cosine (they are denoted by Si and Ci, resp.),
d) Bessel functions (defined by Daniel Bernoulli and generalized to their today form by Friedrich Bessel, usually denoted by J_α and Y_α),
e) zeta function, aka Riemann zeta function (it is denoted by ζ).

We meet some of them later on, in particular when we will discuss so-called gamma distribution and Gauss distribution.

No doubt, the notion 'function' is a central object in the mathematics, its name refers to what is its role in the description of the physical word and in the mathematics itself: a function makes that it functions, it acts, it does its job. For the first time the term 'function' (referring somehow to a mathematical transformation) appeared in Gottfried Wilhelm Leibniz's manuscript *Methodus tangentium inversa, seu de fuctionibus* (1673) and paper *Nova calculi differentialis applicatio & usus, ad multiplicem linearum constructionem, ex data tangentium conditione* (1694), and entered into common use after Leibniz and Johann Bernoulli deliberations, made via the exchange of letters in 1698.

Between functions there are distinguished injections, surjections and bijections; the function f is

a) an **injection**, or an **injective function**, or if never maps distinct elements to the same elements of its codomain; it means that there holds true $\forall_{a,\, b \,\in\, X}\, f(a) = f(b) \Rightarrow a = b$,
b) a **surjection** from X (on)to Y, if for every element $y \in Y$ there exists an element $x \in X$ such that $y = f(x)$; in other words: f is **surjective** if its image is equal to its codomain: $Y = f(X)$,
c) a **bijection**, or **one-to-one correspondence** form X (on)to Y, if it is both injective and surjective.

One can say that a quantity making that a lot of mathematical constructions (or, maybe, all of them) find their applications in descriptions of numerous and often quite different phenomena [3)] is the equivalence relation. A fundamental equivalence in the set theory is an **equinumerosity**: we say that sets A and B are **equinumerous** if there exists a bijection between them.

[3]) Spectacular examples are provided by ODEs (ordinary differential equations). The first order ODE describes the Newtonian cooling process (this thermal phenomena was first considered and reported by Isaac Newton in the paper *Scala graduum caloris*, 1701), the discharge of a condensator (in the electricity) and the exponential growth (serving as a model of the change in the number of people populating the world or in the number of microorganisms with non limited nutrient zasób, or a model of compound interest at a constant rate). The 2nd ODE describes basic phenomena in mechanics (the movement of a mass hanged on the spring) and in the electricity (the change of the current flowing in RLC circuit). A cardioid is a curve traced by a fixed point of the perimeter of a circle rolling around a fixed circle of the same radius, and this mechanical curve is also a caustic (which can be seen on the surface of the coffee in appropriately illuminated cup), its properties are used in so-called cardioid microphones (this kind of unidirectional microphone, probably the most popular, picks up almost exclusively the desired sound while ambient noise is hardly noticeable).

The relation of equinumerosity was introduced by Georg Cantor and resulted, in particular, with Cantor's cardinalities

a) $\aleph_0$ (read: aleph-zero) – the cardinality of the set $\boldsymbol{N} := \{0, 1, 2, 3, \ldots\}$ of natural numbers, the cardinality of any **countable set** (i.e., any collection for which there exists one-to-one correspondence with $\boldsymbol{N}$),
b) c (read: continuum) – the cardinality of the set $\boldsymbol{R}$ of real numbers, the cardinality of any collection equinumerous to $\boldsymbol{R}$ (i.e., any collection for which there exists one-to-one correspondence with $\boldsymbol{R}$).

The above introduces the essential distinction between the sets: any set is
either finite (as, for instance, $\{1, 2, 3, 4\}$),
countable (examples: $\boldsymbol{N}$, $\boldsymbol{N}_0$, $\boldsymbol{Z}$, $\boldsymbol{Q}$)
or **uncountable** (such are the sets $\boldsymbol{R}$ of reals and $\boldsymbol{C}$ of complex numbers).

On sequences

Let's recall that a function defined on countable set is called a **sequence**; usually it is defined on $\boldsymbol{N}$ or $\boldsymbol{N}_0$. The function defined on a finite set (usually: on a finite subset of $\boldsymbol{N}_0$) is also called a sequence, or, more precisely, a **finite sequence**, or n-element sequence, where n is the cardinality of the domain of this function. Commonly, instead of $f(k)$ it is written f_k, k is called the index and f_k is called an element, or a term, no. k, or k-th element, of the sequence (f_k). The type of the f-image determines the adjective the term 'sequence' is accompanied with, so we have, for example, numerical sequences, and they include integer sequences, and real sequences.

Some most frequently used numerical sequences (a_k) are

arithmetical sequence: $a_k = a_0 + k \cdot r$, $k \in \boldsymbol{N}_0$; a_0 and $r \neq 0$ are given reals,
geometrical sequence: $a_k = a_0 \cdot q^k$, $k \in \boldsymbol{N}_0$; a_0 and $q \neq 0$ are given reals,
constant sequence: $(c, c, c, \ldots)$, where c is a constnt number;
it is a particular case of the arithmetics sequence (with the increase $r = 0$) and of the geometric sequence (with the quotient $q = 1$),
zero-one sequence: $(0, 1, 0, 1, 0, 1, \ldots)$,
Bernoulli sequence:

$$(B_k)_{\ k=0,1,2,\ldots} = \left(1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, \ldots\right),$$

Fibonacci sequence:

$$(F_k)_{\ k=0,1,2,\ldots} = (0, 1, 1, 2, 3, 5, 8, \ldots);$$

B_k and F_k are called k-th **Bernoulli number** [4)] and k-th **Fibonacci number**, respectively, and below we put our attention to them (to present some standard treatment of several sequences).

[4)] Bernoulli numbers appear in the formula $\sum\limits_{k=1}^{n} k^m = \frac{1}{m+1} \cdot \sum\limits_{k=0}^{m} \binom{m+1}{k} \cdot B_k \cdot n^{m+1-k}$ referred to as a **Faulhaber formula**. Its left side is the sum

$$1^m + 2^m + 3^m + \ldots + n^m,$$

its right side can be seen as a polynomial in n. So, so we can say that the Faulhaber formula is the representation of the sum of consectutive naturals up to n-th one raised to m-th power in the base composed of powers of n. Explicit expression for $m = 1, 2, 3, \ldots, 17$ was given by Johann Faulhaber in his *Academia Algebrae, darinnen die miraculosische Inventiones, zu den höchsten Cossen weiters continuirt und profitiert werden* (1631). In 1834 Carl Gustav Jakob Jacobi rediscovered the Faulhaber formula and provided its first proof (*De uso legitimo formulae summatoriae Maclaurinianae*). Numbers B_k are named after Jacob Bernoulli who cited Faulhalber's results in *Ars conjectandi* (published in 1713). Coefficients multiplying the products $B_k \cdot n^{m+1-k}$ staying in the right sum in Faulhaber formula are known as binomial coefficients – and we discuss them later on. Bernoulli numbers appear in Euler-Maclaurin formula which is applied to produce so-called Stirling approximation to $n\,!$ – and we present it later on.

Fibonacci numbers are named after Leonardo Fibonacci who introduced them (in fact, starting with $F_1 = F_2 = 1$) in Europe in his *Liber abaci* (1202); in India they were known to Pingala (c.200 BC) and clearly described by Virahanka (c.700 AD).

For every $k \geq 2$ Fibonacci numbers satisfy the formula (referred to as a **Fibonacci recurrence** [5)])

$$F_k = F_{k-1} + F_{k-2},$$

and this formula defines them if $F_0 = 0$, $F_1 = 1$ [6)].

Accordingly to the theory of recursion, to the Fibonacci recurrence there corresponds its algebraic equation (formally created by substitution: the term a_k is replaced by the power r^k)

$$r^k = r^{k-1} + r^{k-2}.$$

Hence, after both sides being multiplicated by r^{k-2}, there is

$$r^2 - r - 1 = 0.$$

This quadratic equation has two zeroes,

$$\frac{1-\sqrt{5}}{2} = -\frac{1}{\Phi} = 1 - \Phi \approx -0.618 \text{ and } \Phi := \frac{1+\sqrt{5}}{2} \approx 1.618,$$

and the general solution, F_k, of the considered recursion is represented as a linear combination

$$F_k = M \cdot \left(-\frac{1}{\Phi}\right)^k + P \cdot \Phi^k = M \cdot \left(\frac{1-\sqrt{5}}{2}\right)^k + P \cdot \left(\frac{1+\sqrt{5}}{2}\right)^k,$$

where M and P are arbitrary constants. Taking into account the initial conditions $F_0 = 0$ and $F_1 = 1$ we easily find that

$$F_k = \frac{1}{\sqrt{5}} \cdot \left\{ \left(\frac{-1}{\Phi}\right)^k + \Phi^k \right\},$$

or

$$F_k = \frac{1}{\sqrt{5}} \cdot \left\{ \left(\frac{\sqrt{5}+1}{2}\right)^k - \left(\frac{\sqrt{5}-1}{2}\right)^k \right\},$$

and it is the explicit formula for k-th Fibonacci number.

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5) A **recurrence**, or a **recursion**, a **recursive formula**, of order r for the sequence $(a_k)_{k=0,1,2,\ldots}$ is the algebraic equation involving numbers necessarily a_k and a_{k+r} (other numbers a_{k+1}, $a_{k+2}, \ldots, a_{k+n-1}$ can also appear). The Fibonacci recurrence is of order 2, it is linear (it says that elements of the sequence at hand are related each to other via the linear combination).

6) With other initial values (i.e., other values for F_0 and F_1) the same recurrence produces other numbers (and, in consequence, another sequence). In particular, with $F_0 = 2$ and $F_1 = 1$ we obtain so-called **Lucas numbers**: 2, 1, 3, 4, 7, 11, 18, 29, … .

From this formula it is easy to see that for k big enough

$$F_k \approx \frac{1}{\sqrt{5}} \cdot \Phi^k = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2} \right)^k .$$

This approximation is very accurate, for instance, for $k = 5, 6, 7, 8, 9$ and 10 it gives values 4.960, 8.025, 12.985, 21.010, 33.994, 55.004.

It is easily checked that the Fibonacci sequence $(F_k)_{k=0,1,2,\ldots}$ is generated [7] by the function $x \to \frac{x}{1-x-x^2}$,

$$\frac{x}{1-x-x^2} = \sum_{k=0}^{\infty} F_k \cdot x^k , \; |x| < \frac{1}{\Phi} .$$

The number Φ (so with the letter starting the name Phidias, a Greek sculptor acting around 450 BC) is called a **golden number**. It was known in the antiguity. The oldest presence of Φ, then called a **sacred ratio**, or a **divine number**, is observed in Egyptian pyramids, the Great Pyramid of Giza (aka Pyramid of Khufu, of Cheops). It was erected of about 591 thousand stone blocks (the biggest one weights 70 tons) around 1560 BC, its height was then 146.5 meters (and is 138.8 m now) and the sloping angle of its sides was 51°50' ≈ arcos(1/Φ). It makes that in the triangle built of the arms of this angle and by the heights one can find a good approximation to the golden number; in so-called Ahmed Papyrus written about 1650 BC it is clearly stated that Hemiunu, the constructor of the Pharaon Cheops Pyramid, preserved the ratio which is today notation is

$$S : B = \Phi,$$

where S is the area of four sides,
B is the area of the pyramid base.

Euclid in his *Elements* (c.300 BC), probably reporting results produced by Theodorus of Cyrene (he lived in 465-398 BC, was a pupil of Protagoras and the tutor of Plato), discussed the division of a segment to parts of which one takes around 0.618 ($\approx \Phi - 1 = 1/\Phi$) of the total length. Moreover, such division observes the double proportion:

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[7] We say that a sequence $(a_k)_{k=0,1,2,3\ldots}$ is generated by the function f, or that f generates this sequence, that f is a (**power**) **generating function**, if a_ks are coefficient in the Maclaurin expansion, i.e., if $f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k$ (and the equality holds true for at least one $x \neq 0$).

If $f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k / k!$, then we say about an **exponentially generated sequence** and about a function generating a sequence exponentially.

For example, the sequence (1, 1, 1, …) is generated by $f(x) = 1/(1-x)$
and is exponentially generated by $f(x) = \exp(x)$.

$$\frac{a+b}{a}=\frac{b}{a}=\Phi,$$

where *a* and *b* are respective longer and shorter part of a given segment (of total length $a + b$). Such division of a given segment is called a **golden section**.

The golden number was also treated around 900 BC by Abu Kamil (he authored *Book of algebra – Kitab fi al-jabr al muquabala*, where, a.o., accepted irrational solutions of equations). The golden section was discussed, in details, by Luca Pacioli in his book *De divina proportione* (1509), which – let it be here mentioned – is rich illustrated with drawings made by Leonardo da Vinci. This great thinker and painter applied the divine proportion in his masterworks (incl. *La ultima cena*, *Ascenzione*), he was probably the one who started to called it as a *sectio aurea* (Latin for the golden section). Exactly 100 years later, in 1509, Johannes Kepler found that

$$\Phi=\lim_{n\to\infty}\frac{F_{n+1}}{F_n};$$

the same limit was independently determined by Albert Girard (announced in *Les Oeuvres mathématiques de Simon Stevin augmentées par Albert Girard* published in 1634, two years after Girard's death).

The **golden proportion** (this is another name for Φ) appears in numerous places, including the proportions in a human body. Moreover, surveys shows that between rectangles the most proportional are considered that observing the golden proportion, i.e., rectangles which sides are related one to another as $\Phi : 1$.

Cardinality of the union of sets

Let's denote:

A – non-empty finite set (i.e., a collection comprising a finite number of elements),

$|A|$ – the cardinality of A, i.e., the number of elements of A,

$U(A) := \{ B : B \subset A \}$ – the family of all subsets of A,

$U_k(A) := \{ B \in U(A) : |B| = k \}$ – the family, whose all elements are all k-element subsets of the set A,

$n\,! := n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ – n factorial defined for any natural n;
additionally, we set $0\,! := 1$.

Obviously, if A and B are finite set sand disjoint (i.e., $A \cap B = \varnothing$), then

$$A \cup B = |A| + |B|.$$

Hence, for any A and B there is

$$\begin{aligned} |A \cup B| &= |(A \setminus B) \cup (A \cap B) \cup (B \setminus A)| = \\ &= |A \setminus B| + |A \cap B| + |B \setminus A| = \\ &= |A \setminus B| + |A \cap B| + |B \setminus A| + |A \cap B| - |A \cap B| = \\ &= |(A \setminus B) \cup (A \cap B)| + |(B \setminus A) \cup (A \cap B)| - |A \cap B| = \\ &= |A| + |B| - |A \cap B|. \end{aligned}$$

This formula for the cardinality of the union (aka the set-theoretic sum) of two finite sets generalizes (applying the mathematical induction) for arbitrary finite union of finite sets in the following way:

we change the sum of cardinalities of all sets by

a) subtracting the sums of cardinalities of all intersections of even numbers of sets

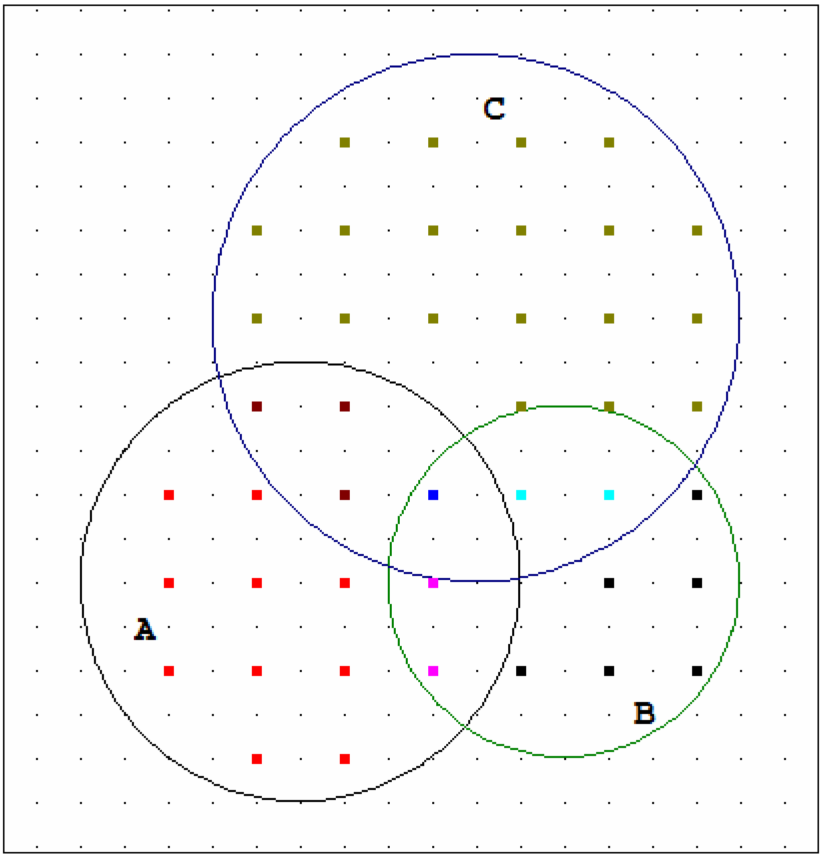
and b) adding the sums of cardinalities of all intersections of odd number of sets;

it is usual to take consecutively all intersections of 2 sets (cardinalities are subtracted), of 3 sets (cardinalities are added), of 4 sets (cardinalities are subtracted), of 5 sets (cardinalities are added) etc. until the cardinality of the common part of all sets is taken into account.

For example, for three finite sets A, B, C the cardinality of their union is:

$$|A \cup B \cup C| = |A| + |B| + |C| - \{ |A \cap B| + |A \cap C| + |B \cap C| \} + |A \cap B \cap C|.$$

This formula can be visualized in the way John Venn proposed in 1880 in the paper *On the diagrammatic and mechanical representation of propositions and reasonings* (and it originated the name 'Venn diagram' advocated by Clarence Lewis in his book *A survey of symbolic logic*, 1918).



Venn diagram for three sets A, B i C showing their unions and intersections;
if A, B and C consists of only dots, there is
$|A| = 16$, $|B| = 11$, $|C| = 25$, $|A \cap B| = 3$, $|A \cap C| = 4$, $|A \cap B \cap C| = 1$

At once it is noticed that for finite and disjoint sets A, B i C there holds

$$|A \cup B \cup C| = |A| + |B| + |C|.$$

Inductively it is show that if sets $A_1, A_2, \ldots, A_n$ are finite and disjoint, then

$$A_1 \cup A_2 \cup \ldots \cup A_n = |A_1| + |A_2| + \ldots + |A_n|.$$

This equality forms the pattern, after which there is formulated the equality

$$A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots = |A_1| + |A_2| + \ldots + |A_n| + \ldots,$$

where there are involved countable many disjoint sets $A_1, A_2, A_3, \ldots$.

Cardinality of the Cartesian product

Let's recall that a Cartesian product of sets A i B is a set

$$A \times B := \{ (a, b) : a \in A; b \in B \}.$$

Obviously, if A and B are finite, then

$$| A \times B | = | A | \cdot | B |.$$

Analogous formula takes place for the Cartesian product of finitely many finite sets, namely it holds

$$| A_1 \times A_2 \times \ldots \times A_n | = | A_1 | \cdot | A_2 | \cdot \ldots \cdot | A_n |.$$